Dynamics 4

Learning Module 4

Lagrange's method for continuous systems

THE UNIVERSITY *of* EDINBURGH
School of Engineering

DYNAMICS 4

LM4 GUIDE 26 27

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Learning Module 4 | Lagrange's method for continuous systems

Estimated time: 18 hours Complete module before: 30 October

1. Start by reading the Course Notes: For this module you should read **Section 2.5.** Take hand-written notes as you read.

2. Attend the lectures and tutorial sessions and, if you need, watch the lecture recordings later. This will help consolidate what you learned. Keep taking notes.

3. Attempt to solve **Questions 4.1 to 4.4** in **Exercise Sheet 4**. Compare your solutions to the numerical solutions provided at the end of each question. Write down any questions you might have.

4. Compare your solutions to the worked-out example solutions to **Questions 4.1 to 4.4**. Discuss any questions you might have in the tutorial sessions. Arrange to meet with your lecturer if you need further clarifications.

5. Be active in the Learn discussion board thread for this Learning Module, which is labeled **continuous_systems**. You can post your own questions and help clarify or answer the questions of others.

6. Have a look at the additional resources provided in the **Resources folder** for Learning Module 4, on Learn.



In this module

Lagrange's method for continuous systems
Rayleigh's method
Single degree of freedom approximation
Multi degree of freedom approximation
Generalised forces in continuous systems

Learning outcomes

Apply virtual work-based methods to dynamical systems, relating between Lagrangian and Newtonian mechanics

Derive equations of motion for dynamic systems from energy-based approaches (e.g. Lagrange's method)

This is a study and learning guide to Learning Module 3. You can find the corresponding lecture recordings and slides on Learn. This module has a dedicated thread in the discussion board on Learn.

Please note that the course time and the recommended *before dates*, are estimated and may differ from person to person. You should not see these as targets but instead as a way to help you plan your semester and the preparation for exams. The estimated time includes going through all the course materials, interaction in the discussion boards and self-study time.

Discrete systems

- Described by a small number of generalised coordinates.
- Rigid bodies.
- Lumped parameter models (e.g. lumped mass).
- Geometry/configuration is irrelevant.

Continuous systems

- Bodies are not rigid... but deformable.
- Dynamic behaviour is governed by continuous elastic deformations.
- Mass is continuously distributed.
- Elasticity and stiffness is also continuously distributed.

Continuous systems

- Most simple continuous structural element: beams
- Examples:
  - Beams and frames in buildings
  - Turbine and wind generator blades
  - Aircraft wings
  - Bridges
  - Ship hulls
  - Pillars, masts, chimneys, etc.



Beams and frames in buildings

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Wind turbine blades

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Aircraft wings

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Bridge structures

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Ship hulls

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Pillars, masts, chimneys, pipework

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Continuous systems

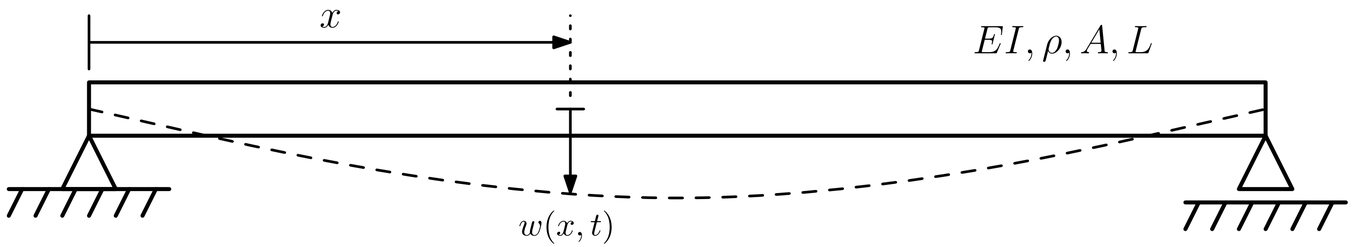
- Specify motion of every single point.
- Leads to an infinite number of degrees of freedom (!)
- ...
- In many applications the dynamic behaviour can be approximated using only a few modes of motion.

Rayleigh's method

- AKA "Assumed modes method".
  1. Reduce infinite number of d.o.f. to a few variables of interest
  2. Consider only the likely / relevant motions of the system
  3. Use engineering intuition
  4. Obtain equations of motion (using Lagrange's method)
  5. Determine lowest natural frequency
- It is an approximate method.
- Results depend on initial guess.

Energy functions (KE, PE)

- Kinetic and potential energy of the whole body.
- Simply supported beam.



Kinetic energy

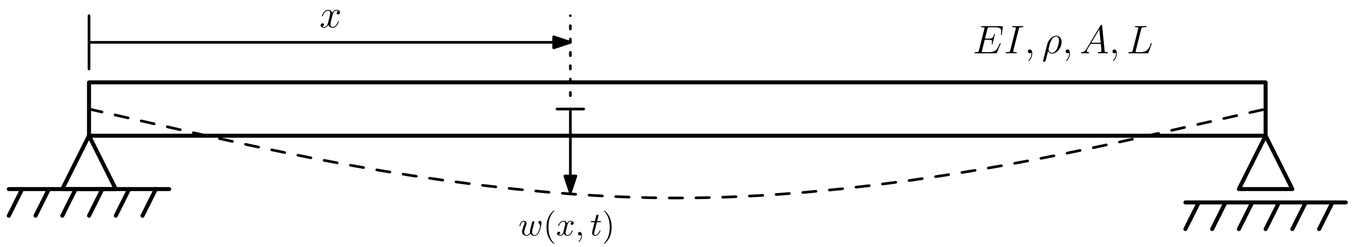
$$T = \int_0^L \frac{\rho A}{2} \left(\frac{\partial w}{\partial t} \right)^2 \mathrm{d}x$$

Potential energy

$$V = \int_0^L \frac{EI}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \mathrm{d}x$$

Energy functions (KE, PE)

- Transverse displacement (*w*) needed to evaluate KE and PE.
- Assume small displacements (i.e. *w* << *L*).



Kinetic energy

$$T = \int_0^L \frac{\rho A}{2} \left(\frac{\partial w}{\partial t} \right)^2 \mathrm{d}x$$

Potential energy

$$V = \int_0^L \frac{EI}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \mathrm{d}x$$

Determine shape of beam

- Describe it using partial differential equations (... later!)
- Develop approximate model using educated guess regarding shape of beam (deflection).
  1. Choose shape function
  2. Must comply with boundary conditions (supports)
  3. Separate space and time dependencies, i.e. assume:

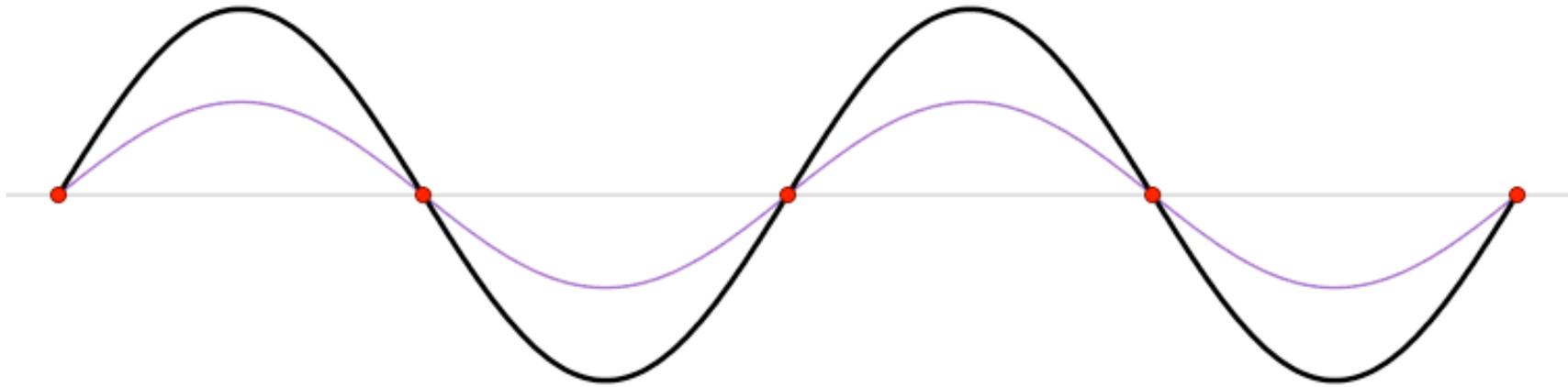
$$w(x,t) = \varphi_1(x) q_1(t)$$

The generalised coordinate (mode amplitude)

The shape function

Determine shape of beam

1. Evaluate Kinetic Energy of the beam
2. Evaluate Potential Energy of the beam
3. Evaluate other energy terms (if they exist)
4. Use Lagrange's equations to derive equations of motion



Determine shape of beam

1. Evaluate Kinetic Energy of the beam:

Kinetic energy →

$$T = \int_0^L \frac{\rho A}{2} \left(\frac{\partial w}{\partial t} \right)^2 \mathrm{d}x = \frac{\rho A}{2} \dot{q}_1^2 \int_0^L [\varphi_1(x)]^2 \mathrm{d}x$$

… yields a constant

2. Evaluate Potential Energy of the beam:

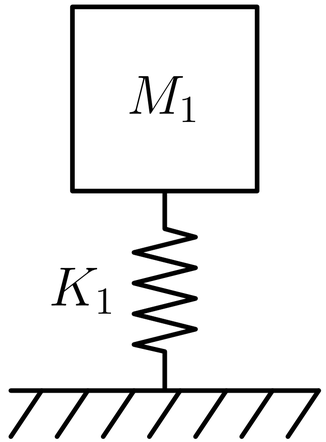
Kinetic energy →

$$V = \int_0^L \frac{EI}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \mathrm{d}x = \frac{EI}{2} q_1^2 \int_0^L \left(\frac{\partial^2 \varphi_1}{\partial x^2} \right)^2 \mathrm{d}x$$

… yields a constant

Determine shape of beam

3. Evaluate other energy terms (if they exist)
4. Use Lagrange's equations to derive equations of motion



- Corresponding equation of motion:

$$M_1\ddot{q}_1 + K_1 q_1 = 0$$

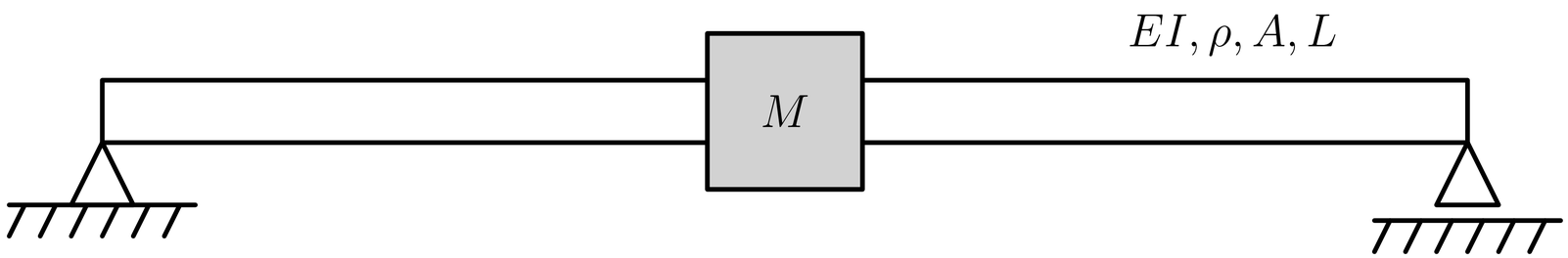
- Solution is sinusoidal vibration with natural frequency:

$$\omega = \sqrt{\frac{K_1}{M_1}}$$

Example (CN)

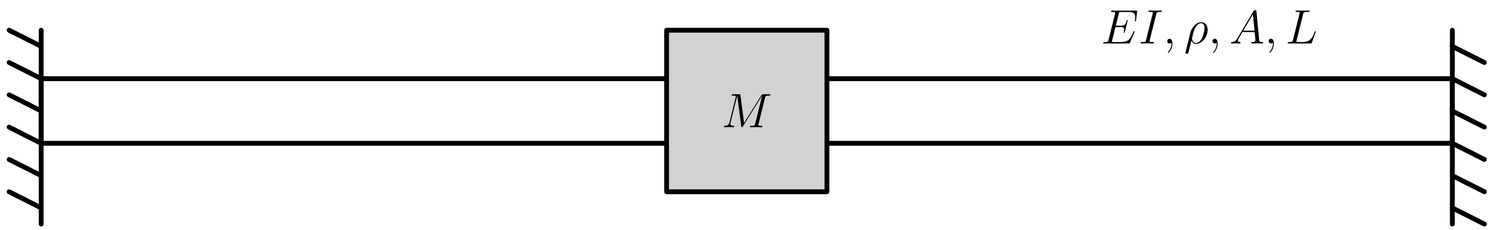
Evaluate the effective mass and stiffness of the beam assuming the following mode shape function:

$$\phi_1(x) = \sin(\pi x / L)$$



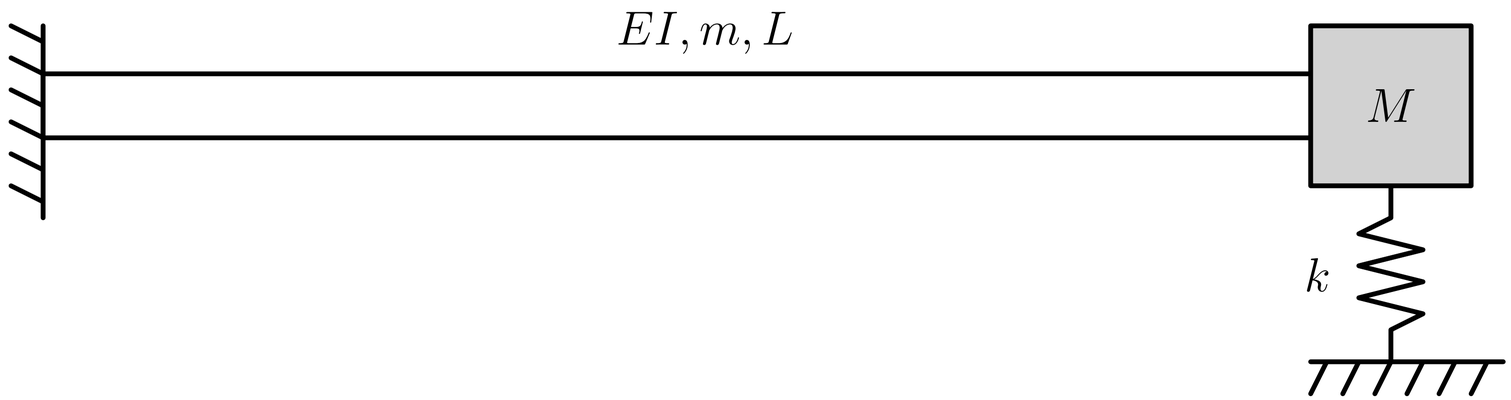
Example (CN)

Evaluate the effective mass and stiffness of the beam. Start by choosing the most convenient mode shape function.



Example (CN)

Derive the equation of motion for the vibration of a clamped beam with a tip mass and spring support.



Multiple degrees of freedom (*n*-d.o.f.)

- Assume more than one shape function.
- Impose boundary conditions.
  - Example of beam deflection:

$$w(x,t) = \sum_{k=1}^{n} \varphi_k(x) q_k(t) \quad + \quad Q_k = \varphi_k(b) F(t)$$

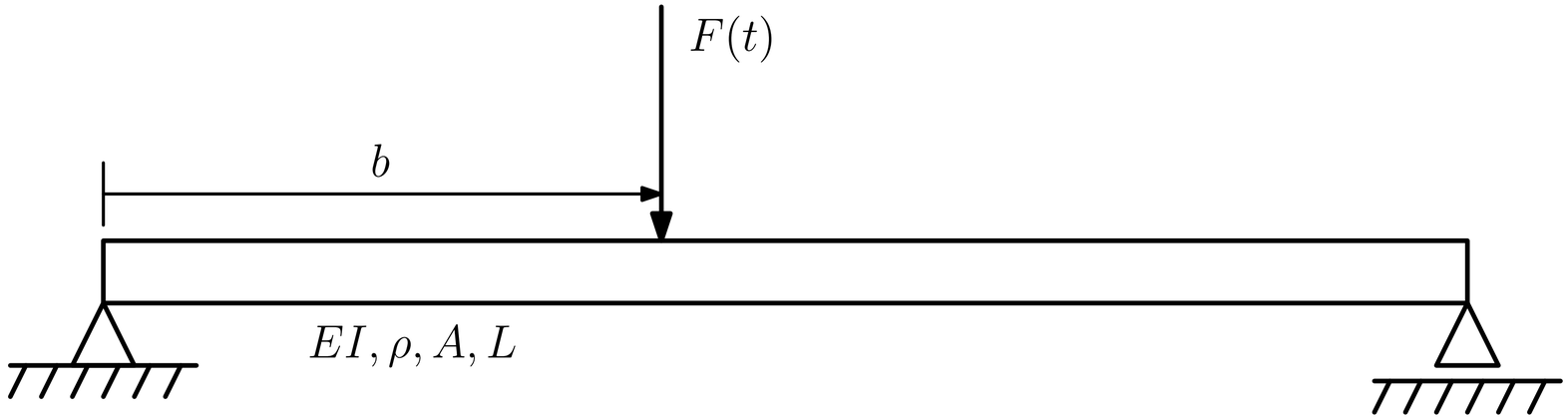
The shape functions

The mode amplitudes

The corresponding generalised forces

Generalised forces

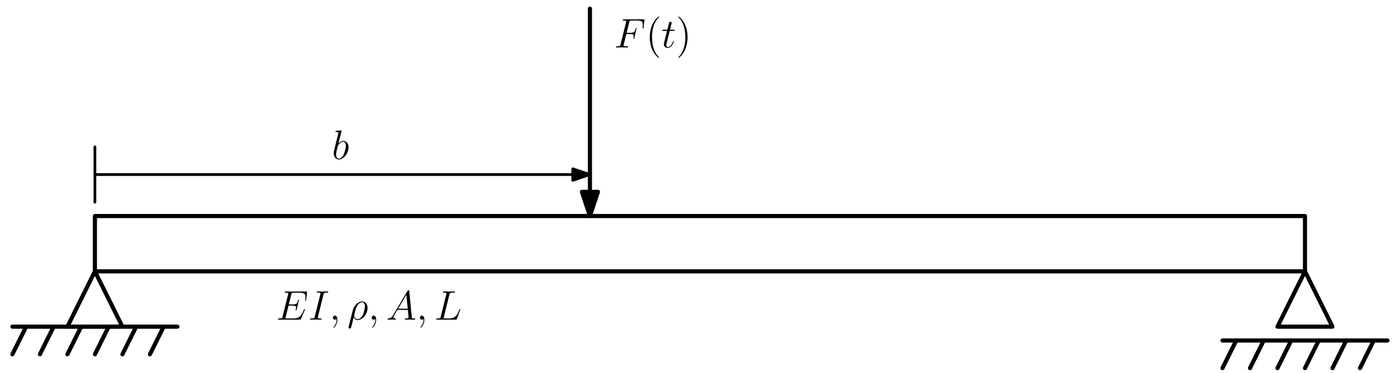
- Forces due to point loads.



Generalised forces

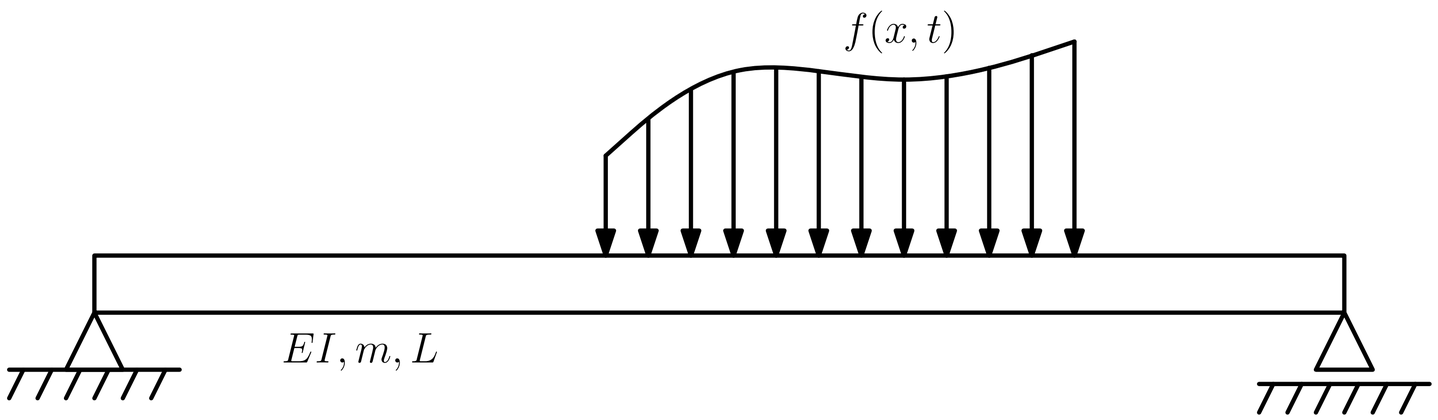
- Forces due to point loads:

  1. Assume one degree of freedom (case shown)
  2. Virtual displacement at $x = b$ is $\delta w = \delta q_1 \varphi_1(b)$
  3. Virtual work done by the force is $\delta q_1 \varphi_1(b) F(t)$
  4. Corresponding generalised force is $\varphi_1(b) F(t)$



Generalised forces

- Forces due to distributed loads.



Generalised forces

- Forces due to distributed loads:

  1. Assume one degree of freedom
  2. Virtual displacement at *x* is $\delta w = \delta q_1 \varphi_1(x)$
  3. Virtual work done by the force is

$$\int_0^L \delta q_1 \varphi_1(x) f(x,t) \mathrm{d}x$$

  4. Corresponding generalised force is

$$Q_1(t) = \int_0^L \varphi_1(x) f(x,t) \mathrm{d}x$$

The function describing the distributed load (in same CS)

Example (CN)

Choose appropriate mode shape functions and derive the equation of motion for the vibration of the flexibly supported beam shown.

